A Note on Rotational Tensor

by

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1. INTRODUCTION

Frequently engineers need to transform a set of vector-valued quantities from one coordinate system into another coordinate system when the connection between these two coordinate systems is a pure rotation of angle θ with respect to some axis vector ***p*** passing thru a unique center of rotation. Especially in the analyses of robotics motion, computational graphics and mechanical mechanism, rotational operations are constantly used. This short note[2] provides a review on how to get the rotation transformation *matrix* promptly. In the classic treatment of rotation transformation, matrix is mostly often used to depict this subject. But the author is convinced that using the mathematical tool of tensor is more geometrically visual than using matrix theory in terms of getting more insight and *feelings* of the rotational *operation* on 3-dimensional vectors. We will start with a brief account of mathematical terminology and preliminary background material on tensor algebra that will be needed in the later derivation. We will only discuss a rotation that map a 3-dimensional vector to another 3-dimensional vector. Any hyper-rotation that involves higher than 3-dimensional vectors is not our interest herein.

2. MATHEMATICAL PRELIMINARY[3]

The tensor product $\mathbf{a} \otimes \mathbf{b}$[4] of any two vectors **a** and **b** is the *second-order* tensor[5] defined by

$$(\mathbf{a} \otimes \mathbf{b})\, \mathbf{u} = \mathbf{a}\, (\mathbf{b} \cdot \mathbf{u}) \qquad (2\text{-}1)$$

for every vector **u**. The dot represents an inner product operation of two vectors. It is also necessary for later usage to introduce a 3^{rd} *order* tensor ($\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}$) that has the following defined operations:

$$\begin{aligned} &(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c})\boldsymbol{u} = \boldsymbol{a} \otimes \boldsymbol{b}\, (\boldsymbol{c} \cdot \boldsymbol{u}) \\ &(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c})\, [\boldsymbol{u} \otimes \boldsymbol{v}] = \boldsymbol{a}\, (\boldsymbol{b} \cdot \boldsymbol{u})(\boldsymbol{c} \cdot \boldsymbol{v}) \end{aligned} \qquad (2\text{-}2)$$

Let $\{\mathbf{e}_j\}$ (j = 1, 2, 3) be any right-handed orthonormal basis of vectors. Then, any vector **a** and any second-order tensor **A** may be represented as[6]:

$$\begin{aligned} &\boldsymbol{a} = a_j \boldsymbol{e}_j \\ &\boldsymbol{A} = A_{jk} (\boldsymbol{e}_j \otimes \boldsymbol{e}_k) \\ &A_{jk} = \boldsymbol{e}_j \cdot \boldsymbol{A} \boldsymbol{e}_k \end{aligned} \qquad (2\text{-}3)$$

The transpose $\mathbf{A}^T$ of a 2nd order tensor **A** is defined as:

$$\boldsymbol{A}^T = A_{jk} (\boldsymbol{e}_k \otimes \boldsymbol{e}_j) \qquad (2\text{-}4)$$

Thus, it readily follows that for all vectors **u** and **v**,

$$\begin{aligned} &\boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{A}^T \boldsymbol{v} \\ &(\boldsymbol{a} \otimes \boldsymbol{b})^T = (\boldsymbol{b} \otimes \boldsymbol{a}) \end{aligned} \qquad (2\text{-}5)$$

It is observed that

$$\begin{aligned} &\boldsymbol{A}\boldsymbol{B}\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (\boldsymbol{A}\boldsymbol{B})^T \boldsymbol{v} \\ &\boldsymbol{A}\boldsymbol{B}\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{B}\boldsymbol{u} \cdot \boldsymbol{A}^T \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{B}^T \boldsymbol{A}^T \boldsymbol{v}, \text{ so} \\ &(\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T \boldsymbol{A}^T \end{aligned} \qquad (2\text{-}6)$$

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[1] S. Tseng is a Mechanical Engineer of Bastion Technologies, Houston, Texas. The contents of this note are mainly based on [1].

[2] There are extensive existing literature on the rotational algebra such as [1], [4] thru [10].

[3] If the reader is interested in knowing more detail, Chapters 1 and 2 of [4] provide a good reference.

[4] Bold lower case is used to represent a vector; and bold upper case represents a tensor of order equal to or higher than two.

[5] A vector is generally deemed as a 1st order tensor. And scalars are zero order tensors.

[6] All indices have a range 1, 2, 3 and the convention of summations over repeated indices is employed.

Similarly, it is readily proven:

$$(\boldsymbol{A}^T)^T = \boldsymbol{A} \tag{2-7}$$

The identity tensor **I** is defined as an operation that will map any vector **u** to itself; and has a representation:

$$\boldsymbol{I} = \delta_{jk}(\boldsymbol{e}_j \otimes \boldsymbol{e}_k) = (\boldsymbol{e}_j \otimes \boldsymbol{e}_j). \tag{2-8}$$

where δ_{jk} is 1 when j=k and zero otherwise. The 3rd order alternate tensor **ε** is defined as:

$$\boldsymbol{\varepsilon} = \varepsilon_{jkl}(\boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l) \tag{2-9}$$

where ε_{jkl} has value +1 if jkl is an even permutation of 123, -1 if jkl is an odd permutation of 123, and zero otherwise. For any vectors ***a*** and ***b***, we have:

$$\boldsymbol{\varepsilon}[\boldsymbol{a} \otimes \boldsymbol{b}] = \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{\varepsilon}[\boldsymbol{a}]\boldsymbol{b} \tag{2-10}$$

The axial vector a of the skew tensor $(\boldsymbol{u} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u})$ is defined as:

$$\boldsymbol{a} = -\frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{u} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u}) = -(\boldsymbol{u} \times \boldsymbol{v}) \tag{2-11}$$

Let ***p*** ***q*** and ***r*** be a set of right-handed orthonormal vectors. With some persistence in employing the tensor algebra introduced above, the following identities can be proven:

$$\begin{aligned}
&\boldsymbol{\varepsilon}[\boldsymbol{p}] = (\boldsymbol{q} \otimes \boldsymbol{r} - \boldsymbol{r} \otimes \boldsymbol{q}) \\
&\boldsymbol{\varepsilon}[\boldsymbol{p}]\,\boldsymbol{\varepsilon}[\boldsymbol{p}] = (\boldsymbol{p} \otimes \boldsymbol{p}) - \boldsymbol{I} \qquad\qquad (2\text{-}12) \\
&\boldsymbol{\varepsilon}[\boldsymbol{p}]\,\boldsymbol{\varepsilon}[\boldsymbol{p}]\,\boldsymbol{\varepsilon}[\boldsymbol{p}] = \boldsymbol{\varepsilon}[\boldsymbol{p}]\{(\boldsymbol{p} \otimes \boldsymbol{p}) - \boldsymbol{I}\} = -\boldsymbol{\varepsilon}[\boldsymbol{p}]
\end{aligned}$$

The determinant of the 2nd order tensor A is defined as:

$$\det(\boldsymbol{A}) = \boldsymbol{Au} \cdot (\boldsymbol{Av} \times \boldsymbol{Aw}) \big/ \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) \tag{2-13}$$

When **u**, **v** and **w** are linearly independent vectors, the magnitude of the scalar[7] $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ represents the volume [8] of the parallelepiped determined by **u**, **v** and **w**; and is frequently denoted as [**u**, **v**, **w**]. Thus, [**Au**, **Av**, **Aw**] = $\mathbf{Au} \cdot (\mathbf{Av} \times \mathbf{Aw})$ represents the volume of transformed parallelepiped obtained by applying operation **A** on the parallelepiped determined by **u**, **v** and **w**. So if

$$\boldsymbol{u}' = \boldsymbol{Bu}, \quad \boldsymbol{v}' = \boldsymbol{Bv}, \quad \boldsymbol{w}' = \boldsymbol{Bw}$$

the following identities follow:

$$[\boldsymbol{Bu}, \boldsymbol{Bv}, \boldsymbol{Bw}] = \det(\boldsymbol{B})\,[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$$

$$[\boldsymbol{Au}', \boldsymbol{Av}', \boldsymbol{Aw}'] = \det(\boldsymbol{A})\,[\boldsymbol{u}', \boldsymbol{v}', \boldsymbol{w}']$$

$$[\boldsymbol{ABu}, \boldsymbol{ABv}, \boldsymbol{ABw}] = \det(\boldsymbol{A})\,[\boldsymbol{Bu}, \boldsymbol{Bv}, \boldsymbol{Bw}] = \det(\boldsymbol{A}) \cdot \det(\boldsymbol{B})\,[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$$

$$[\boldsymbol{ABu}, \boldsymbol{ABv}, \boldsymbol{ABw}] = \det(\boldsymbol{AB})\,[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \tag{2-14}$$

It is observed from the last identity of (2-14) that:

$$\therefore \det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \cdot \det(\boldsymbol{B}) \tag{2-15}$$

3. ROTATIONAL TENSOR

Before the rotational tensor is discussed, the more general orthogonal tensor is studied first. A tensor Q is orthogonal if and only if it preserves inner products of any two vectors **u** and **v** as shown below:

$$\begin{aligned}
&\boldsymbol{Qu} \cdot \boldsymbol{Qv} = \boldsymbol{u} \cdot \boldsymbol{v} \\
&\boldsymbol{u} \cdot \boldsymbol{Q}^T\boldsymbol{Qv} = \boldsymbol{u} \cdot \boldsymbol{Iv}
\end{aligned} \tag{3-1}$$

Thus, if a 2nd order tensor Q is orthogonal, we have:

$$\begin{aligned}
&\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I} \\
&\det(\boldsymbol{Q}) = \pm 1
\end{aligned} \tag{3-2}$$

From $(3\text{-}2)_1$, it follows:

$$\boldsymbol{Q}^T(\boldsymbol{Q} - \boldsymbol{I}) = \boldsymbol{I} - \boldsymbol{Q}^T = -(\boldsymbol{Q} - \boldsymbol{I})^T \tag{3-3}$$

Consequently, if ***Q*** is an orthogonal tensor, from (3-3), we have

$$\det(\boldsymbol{Q} - \boldsymbol{I}) = 0.$$

Therefore, ***Q*** has a unit eigenvalue. Let the associated eigenvector be ***p***, it follows that

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[7] This scalar is usually called the triple product of three vectors.

[8] Mathematically, this volume value may be negative depending on how the vectors **u**, **v**, **w** are oriented with respect to one another.

$$\boldsymbol{Q}\,\boldsymbol{p} = \boldsymbol{p}, \quad \boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{p} = \boldsymbol{p}. \tag{3-4}$$

Let $\boldsymbol{Q}$ be a proper orthogonal tensor (i.e. det($\boldsymbol{Q}$) = 1 and $\boldsymbol{Q}$ is a rotational tensor). Forming a right-handed orthonormal basis {$\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$} such that

$$\boldsymbol{p} = \boldsymbol{q} \times \boldsymbol{r} \tag{3-5}$$

Therefore, we have

$$\boldsymbol{p} \cdot \boldsymbol{q} = \boldsymbol{q} \cdot \boldsymbol{p} = 0, \quad \boldsymbol{p} \cdot \boldsymbol{r} = \boldsymbol{r} \cdot \boldsymbol{p} = \boldsymbol{0} \tag{3-6}$$

From (2-3), (3-4) and (3-6), we find that

$$\begin{aligned}
Q_{pq} &= \boldsymbol{p} \cdot \boldsymbol{Qq} = \boldsymbol{q} \cdot \boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{p} = \boldsymbol{q} \cdot \boldsymbol{p} = 0\\
Q_{qp} &= \boldsymbol{q} \cdot \boldsymbol{Qp} = \boldsymbol{q} \cdot \boldsymbol{p} = 0\\
Q_{pr} &= \boldsymbol{p} \cdot \boldsymbol{Qr} = \boldsymbol{r} \cdot \boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{p} = \boldsymbol{r} \cdot \boldsymbol{p} = 0\\
Q_{rp} &= \boldsymbol{r} \cdot \boldsymbol{Qp} = \boldsymbol{r} \cdot \boldsymbol{p} = 0
\end{aligned} \tag{3-7}$$

Thus, the component form of Q in terms of the basis is

$$\boldsymbol{Q} = \boldsymbol{p} \otimes \boldsymbol{p} + Q_{qq}\,\boldsymbol{q} \otimes \boldsymbol{q} + Q_{rr}\,\boldsymbol{r} \otimes \boldsymbol{r} + Q_{qr}\,\boldsymbol{q} \otimes \boldsymbol{r} + Q_{rq}\,\boldsymbol{r} \otimes \boldsymbol{q} \tag{3-8}$$

It is obvious from (3-1)$_1$ that the orthogonal transformations preserve the length of any vector as well as the angle between any pair of vectors in the space. Therefore, from (3-7)$_{1,3}$ and (3-1) it follows that $\boldsymbol{Qq}$ and $\boldsymbol{Qr}$ are orthogonal to each other, and are on the same plane which is normal to $\boldsymbol{p}$ as shown in Figure 3-1. From Figure 1,

$$\begin{aligned}
Q_{qq} &= \boldsymbol{q} \cdot \boldsymbol{Q}\,\boldsymbol{q} = \cos\boldsymbol{\theta}\\
Q_{qr} &= \boldsymbol{q} \cdot \boldsymbol{Q}\,\boldsymbol{r} = \cos\left({}^{\pi}/_{2} + \boldsymbol{\theta}\right) = -\sin\boldsymbol{\theta}\\
Q_{rq} &= \boldsymbol{r} \cdot \boldsymbol{Q}\,\boldsymbol{q} = \cos\left({}^{\pi}/_{2} - \boldsymbol{\theta}\right) = \sin\boldsymbol{\theta}\\
Q_{rr} &= \boldsymbol{r} \cdot \boldsymbol{Q}\,\boldsymbol{r} = \cos\boldsymbol{\theta},
\end{aligned} \tag{3-9}$$

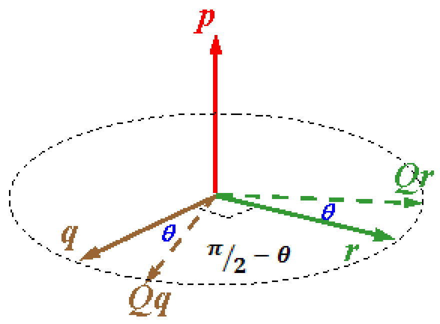


Figure 1

where θ is the angle between $\boldsymbol{Qq}$ and $\boldsymbol{q}$ (i.e., the same angle between $\boldsymbol{Qr}$ and $\boldsymbol{r}$).

By (2-12) and (3-9), (3-8) becomes

$$\begin{aligned}
\boldsymbol{Q} &= \boldsymbol{p} \otimes \boldsymbol{p} + \cos\theta\,(\boldsymbol{q} \otimes \boldsymbol{q} + \boldsymbol{r} \otimes \boldsymbol{r}) - \sin\boldsymbol{\theta}\,(\boldsymbol{q} \otimes \boldsymbol{r} - \boldsymbol{r} \otimes \boldsymbol{q})\\
&= \boldsymbol{I} - (\boldsymbol{1} - \cos\theta)\,(\boldsymbol{q} \otimes \boldsymbol{q} + \boldsymbol{r} \otimes \boldsymbol{r}) - \sin\theta\,(\boldsymbol{q} \otimes \boldsymbol{r} - \boldsymbol{r} \otimes \boldsymbol{q})\\
&= \boldsymbol{I} - \sin\theta\;\boldsymbol{\varepsilon}[\boldsymbol{p}] + (1 - \cos\theta)(\boldsymbol{\varepsilon}[\boldsymbol{p}])^{\boldsymbol{2}}
\end{aligned} \tag{3-10}$$

4. GEOMETRIC INTERPRETATION

Since it is a rotation, our instinct tells us formula (3-10) must have some geometric meaning. It is observed from (2-12)$_2$ that $-\boldsymbol{\varepsilon}[\boldsymbol{p}]\boldsymbol{\varepsilon}[\boldsymbol{p}]$ is actually a projection operator that maps any vector $\boldsymbol{r}$ to its image vector on the plane with a unit normal vector $\boldsymbol{p}$ as shown below:

$$(\boldsymbol{I} - \boldsymbol{p} \otimes \boldsymbol{p})\boldsymbol{r} = \boldsymbol{r} - (\boldsymbol{p} \cdot \boldsymbol{r})\,\boldsymbol{p} \tag{4-1}$$

where $(\boldsymbol{p} \cdot \boldsymbol{r})\boldsymbol{p}$ is the component vector of $\boldsymbol{r}$ along normal $\boldsymbol{p}$. And from (2-10), it is also observed that

$$-\,\boldsymbol{\varepsilon}[\boldsymbol{p}]\,\boldsymbol{r} = \boldsymbol{p} \times \boldsymbol{r} \tag{4-2}$$

Thus as shown in Figure 2, the rotation tensor of (3-10) will map any vector $\boldsymbol{r}$ into $\boldsymbol{r'}$ as follows:

$$\boldsymbol{r}' = \boldsymbol{Q}[\boldsymbol{r}] = \boldsymbol{r} + \sin\theta\ \boldsymbol{p}\times\boldsymbol{r} - (1-\cos\theta)\cdot(\boldsymbol{r}-(\boldsymbol{p}\cdot\boldsymbol{r})\,\boldsymbol{p}) \qquad (4\text{-}3)$$

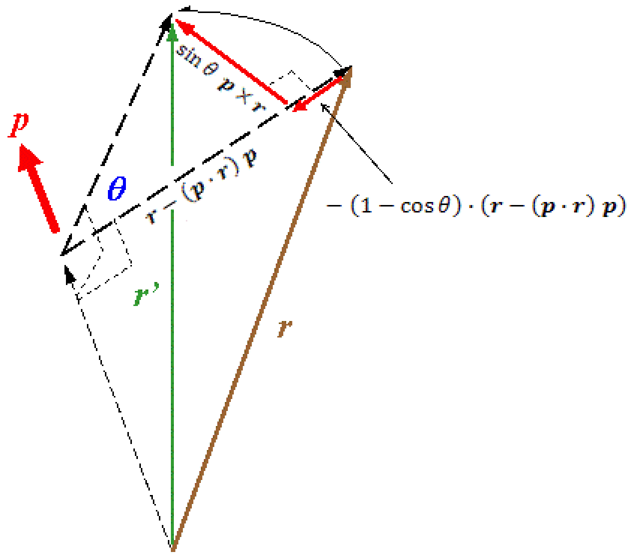


Figure 2

5. EXAMPLE

A MathCAD algorithm is used to illustrate the application of formula (3-10) derived above. Notice that when translating a tensor equation into the matrix equivalence, the components of the matrix are exactly the transpose of the tensor components.

From (2-10), the operation of $-\sin\theta\ \boldsymbol{\varepsilon}[\boldsymbol{p}]$ on any vector $\boldsymbol{v}$ is equivalent to $\sin\theta\ \boldsymbol{p}\times\boldsymbol{v}$. The equivalent matrix representation of this operation is:

$$-\sin\theta\ \boldsymbol{\epsilon}[\boldsymbol{p}] \sim +\sin\theta\begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$$

where p_1, p_2 p_3 are the components of the unit rotational axis vector $\boldsymbol{p}$. From $(2\text{-}12)_2$, the equivalent matrix representation of the operation of $(\boldsymbol{\varepsilon}[\boldsymbol{p}])^2$ on any vector is:

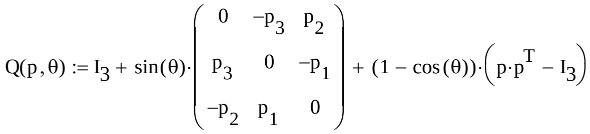
$$(\boldsymbol{\varepsilon}[\boldsymbol{p}])^2 \sim \boldsymbol{p}\cdot\boldsymbol{p}^T - \boldsymbol{I}$$

where $\boldsymbol{I}$ is a 3 x 3 identity matrix; and $\boldsymbol{pp}^{\mathrm{T}}$ is a matrix product of the column matrix representation of $\boldsymbol{p}$ and its transpose (i.e. row matrix).

The rotational matrix representation for a rotational tensor which imposes a rotation of θ about a unit axial direction $\boldsymbol{p}$ can be written as follows:

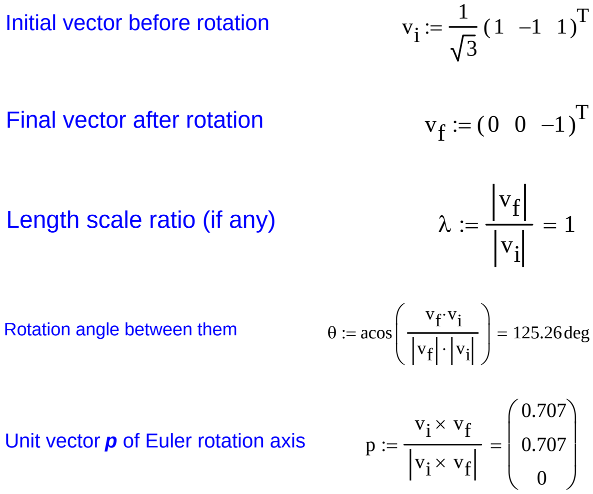
$$[Q] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta\begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} + (1-\cos\theta)\begin{bmatrix} p_1p_1-1 & p_1p_2 & p_1p_3 \\ p_2p_1 & p_2p_2-1 & p_2p_3 \\ p_3p_1 & p_3p_2 & p_3p_3-1 \end{bmatrix} \qquad (5\text{-}1)$$

An example of using MathCAD to define (5-1) is



$$Q(p,\theta) := I_3 + \sin(\theta)\cdot\begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix} + (1-\cos(\theta))\cdot\left(p\cdot p^{T} - I_3\right)$$

Two vectors are given. To find the rotational matrix that will rotate vector $\mathbf{v}_{\mathrm{i}}$ to vector $\mathbf{v}_{\mathrm{f}}$. The unit axial vector $\boldsymbol{p}$ is along the normal of the plane spanned by these two given vectors.



Initial vector before rotation $\quad v_i := \frac{1}{\sqrt{3}}(1\ \ -1\ \ 1)^T$

Final vector after rotation $\quad v_f := (0\ \ 0\ \ -1)^T$

Length scale ratio (if any) $\quad \lambda := \frac{|v_f|}{|v_i|} = 1$

Rotation angle between them $\quad \theta := \mathrm{acos}\left(\frac{v_f\cdot v_i}{|v_f|\cdot|v_i|}\right) = 125.26\,\mathrm{deg}$

Unit vector ***p*** of Euler rotation axis $\quad p := \frac{v_i\times v_f}{|v_i\times v_f|} = \begin{pmatrix} 0.707 \\ 0.707 \\ 0 \end{pmatrix}$

Rotation matrix $$R := Q(p,\theta) = \begin{pmatrix} 0.211 & 0.789 & 0.577 \\ 0.789 & 0.211 & -0.577 \\ -0.577 & 0.577 & -0.577 \end{pmatrix}$$

Verification check $$R\cdot v_i\cdot\lambda = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Figure 3 provides a visual proof of the above example problem. The diagonal edge of one wedge represents vector $\mathbf{v}_i$; and the diagonal edge of the other wedge represents vector $\mathbf{v}_f$. And the red arrow represents unit axial rotational vector $\boldsymbol{p}$.

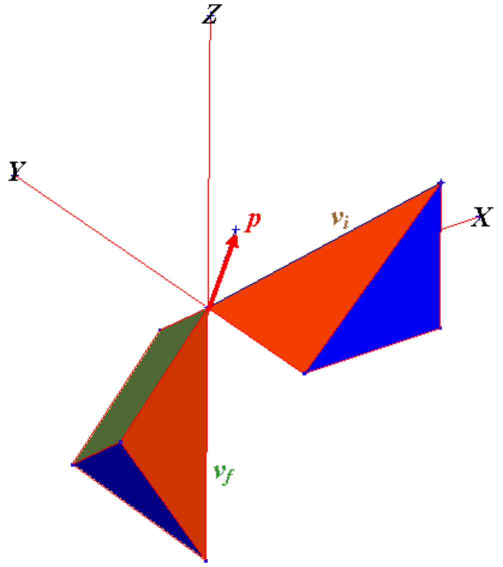


Figure 3

6. REFERENCE

[1] Casey, J., 1983, Class notes of the course of "Geometry of Motions" taught by Professor Casey.

[2] Casey, J. and Lam, V., 1986, "A Tensor Method for the Kinematical Analysis of Systems of Rigid Bodies", *Mechanism and Machine Theory*, Vol. **21**, No. 1, pp. 87-97.

[3] Naghdi, P.M. and Vongsarnpigoon, L., 1982, "Small Strain Accompanied by Moderate Rotation", *Archive for Rational Mechanics and Analysis*, Vol. **80**, No. 3, 1982, pp. 263-294.

[4] Gurtin, M., An Introduction to Continuum Mechanics, 1981, Academic Press.

[5] Faux, D. and Pratt, M.J., Computational Geometry for Design and Manufacture, 1979, by Ellis Horwood.

[6] Arthur Mayer, 1960, "ROTATIONS AND THEIR ALGEBRA", *SIAM REVIEW*, Vol. **2**, No. 2, pp.77-122.

[7] Millard Beatty, 1966, "Kinematics of Finite Rigid-Body Displacements", *American Journal of Physics*, Vol. **34**, pp. 949-954.

[8] Robert Hilborn, 1972, "A Note on Euler Angle Rotations", *American Journal of Physics*, Vol. **40**, pp. 1036-1037.

[9] Millard Beatty, 1977, "Vector Analysis of Finite Rigid Rotations", *Journal of Applied Mechanics*, Vol. **44**, pp. 1036-1037.

[10] Kane, T.R., and Levinson, D.A., 1978, "Successive Finite Rotations", *ASME Journal of Applied Mechanics*, Vold. **45**, pp. 945-946.

[11] Leubner, C., 1979, "Coordinate-free rotation operator", *American Journal of Physics*, Vol. **47**, No. 7, pp. 727-729.

[12] Leubner, C., 1981, "Correcting a Widespread error concerning the angular velocity of a rotating rigid body", *American Journal of Physics*, Vol. **49**, No. 3, pp. 232-234.